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LEVEL RECOVERY CURVE IN THE RELAXATION THEORY OF FILTRATION

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A problem on the level recovery curve in the relaxation theory of filtration is considered when there is a continuous spectrum of internal relaxation times. An asymptotics at large times is found as a functional of a relaxation kernel. An explicit expression with two additional parameters characterizing the relaxation kernel is calculated for a power spectrum of internal relaxation processes in a rock-saturating fluid system.

Darcy's law in the linear theory of filtration is valid only for processes where characteristic times of change in macroscopic parameters (for example, of a pressure gradient) are much larger than the characteristic internal relaxation time in a porous medium-saturating fluid system on a microlevel. Otherwise, it is necessary to use generalizations of Darcy's law by the relaxation theory of filtration that were suggested in [1-6]. There are situations when a relaxation law of filtration can be strictly derived from the kinetic theory [7].

Internal relaxation processes can be manifested in nonstationary hydrodynamic investigations of wells; therefore, on interpretation they should be taken into account along with such factors affecting the dynamics as the geological structure of a well-botton zone. Previously, the theoretical results were concerned with the form of the pressure recovery curve (PRC) over an initial section [8] and with the asymptotics of the PRC at large times for discrete and continuous spectra of internal relaxation times [9, 10].

In the present work within the framework of relaxation isothermal theory of filtration we investigate the problem on a level recovery curve (LRC) in a vertical well for a case of a single-phase slightly compressible liquid in a homogeneous isotropic collector.

For the arbitrary time function $f = f(t)$ we denote the Fourier transformation by the symbol $f_F = f_F(\omega)$

$$f_F(\omega) = \int_{-\infty}^{+\infty} \exp(-i\omega t) f(t)\, dt .$$

In the relaxation theory of filtration Darcy's law is generalized in the following manner [1-6]:

$$u^i(t_0, x^j) = -k\mu^{-1} \int_{-\infty}^{+\infty} K(t_0 - t) \frac{\partial G}{\partial x^i}(t, x^j)\, dt , \quad (1)$$

where $G = p + \rho U$; i, j run over the values 1, 2, 3, which correspond to Cartesian coordinates x^i.

The kernel $K = K(t)$ describes internal relaxation processes in the porous medium-saturating fluid system. For the kernel some conditions are fulfilled:

1) $K(t)$ is a nonnegative monotonically decreasing function that has the dimensionality t^{-1};

2) $\int_{-\infty}^{+\infty} K(t)dt = 1$ is the condition for reduction of (1) to Darcy's law for slow processes;

3) $K(t) = 0$ with $t < 0$ (causality); $0 < K(0) < +\infty$ is the condition of signal-velocity finiteness [11];

4) $\mathrm{Re}\, K_F(\omega) > 0$ with $\mathrm{Im}\, \omega \le 0$ is the dissipativity condition [4, 6].

By virtue of condition (3) in accordance with a Paley–Wiener theorem [12] the function $K_F = K_F(\omega)$ in the lower half-plane of the complex plane is holomorphic.

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From condition 2) it follows that

$$K_F(0) = 1 . \tag{2}$$

For the relaxation kernel we take an expression that corresponds to the continuous spectrum of purely dissipative internal relaxation processes:

$$K(t) = \int_0^{+\infty} A(\tau)\,\tau^{-1} \exp(-t/\tau)\, d\tau , \tag{3}$$

where $A(\tau)$ is a smooth nonnegative function. In the Fourier transform, expression (3) takes the form

$$K_F(\omega) = \int_0^{+\infty} A(\tau)\,(1 + i\tau\omega)^{-1}\, d\tau . \tag{4}$$

Equations (2) and (4) yield the normalizing equality

$$1 = \int_0^{+\infty} A(\tau)\, d\tau . \tag{5}$$

In addition, the integral convergence results from condition 3)

$$k_1 = \int_0^{+\infty} \tau^{-1} A(\tau)\, d\tau < +\infty . \tag{6}$$

Relations (3)-(6) are suffice to carry out conditions 1)-4) for the relaxational kernel. From expression (4) it follows that the function $K_F(\omega)$ is holomorphic with a cut along the beam $\mathrm{Re}\,\omega = 0$, $\mathrm{Im}\,\omega > 0$. Using a Sokhotskii–Plemel formula, it is simple to calculate the function $K_F(\omega)$ on the cut shores:

$$K_{F+} = K_F(iy + \varepsilon) = L_1(y) - i\pi L_2(y) ,$$
$$K_{F-} = K_F(iy - \varepsilon) = L_1(y) + i\pi L_2(y) , \tag{7}$$

$$L_1(y) = \text{V.p.} \int_0^{+\infty} z^{-1} A(z^{-1})\,(z - y)^{-1}\, dz ,$$

$$L_2(y) = y^{-1} A(y^{-1}) .$$

Here and below, $y > 0$; ε is an infinitesimal positive number.

Now we consider a linear problem on the LRC in a cylindrically symmetric statement (i.e., for a vertical well) in the case where there is only one productive layer. The pressure field dynamics is determined by the integro-differential equation [10]

$$\frac{\partial}{\partial t} p(t_0, r) = \kappa \int_{-\infty}^{+\infty} K(t_0 - t)\,\Delta p(t, r)\, dt , \tag{8}$$

where $\kappa = kE/(m\mu)$; $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r$; $E = (E_1^{-1} + (m^{-1} - 1)E_2^{-1})^{-1}$. The parameter r changes within the limits $r_1 \le r \le r_2$.

The pressure on the well bottom is determined by liquid-column dynamics

$$\left.\frac{\partial p}{\partial t}\right|_{r=r_1} = \nu\,(q - Q) . \tag{9}$$

Here $q = q(t) = \lambda \int\limits_{-\infty}^{+\infty} K(t_0 - t)\frac{\partial}{\partial r}p(t, r_1)dt$, $\lambda = 2\pi r_1 hk\rho\mu^{-1}$, $\nu = S^{-1}g$, $Q = Q(t)$ is a given function that characterizes the mass removal of liquid from the well.

On the supply contour the pressure equals the given bed pressure p_{bed}

$$p\,(t, r_2) = p_{\text{bed}}\,. \tag{10}$$

Hereafter, we employ a system of measurement units, in which the following equalities are fulfilled:

$$\kappa = r_1 = 1\,. \tag{11}$$

The quantity κ has the dimensionality l^2/t (l is the length), therefore condition (11) fixes the unit length and unit time.

We will solve problem (8)-(10) for the case when the selection function $Q = Q(t)$ at the instant of time $t = 0$ changes over abruptly from one constant value to another:

$$Q\,(t) = \begin{cases} Q_0\,, & t \le 0\,, \\ Q_1\,, & t > 0\,. \end{cases}$$

The process when $Q_1 = 0$ is usually called the level recovery.

We introduce a new unknown function

$$\Phi = \Phi\,(t, r) = p\,(t, r) - p_{\text{bed}} - \lambda^{-1}\,Q_0 \ln\,(r/r_2)\,.$$

The function $\varphi(t) = \Phi(t, 1)$ sets to zero at negative times, whereas at positive times it characterizes the change in the bottom pressure after the change-over of the regime. In the case of small debits, where hydrodynamic effects in a well shaft can be neglected, this function is linearly related to the change in the liquid column.

Performing the Fourier transformation in Eqs. (8)-(10), we obtain the second-order ordinary differential equation

$$(\Delta - \alpha^2)\,\Phi_{\text{F}} = 0 \tag{12}$$

with boundary conditions

$$\left(i\omega\Phi_{\text{F}} - \xi K_{\text{F}}\,\frac{\partial}{\partial r}\,\Phi_{\text{F}}\right)\Bigg|_{r=1} = \eta\,(i\omega + \varepsilon)^{-1}\,, \quad \Phi_{\text{F}}\,|_{r=r_2} = 0\,, \tag{13}$$

where $\xi = \nu\lambda$; $\eta = \nu(Q_1 - Q_0)$; the complex function $\alpha = \alpha(\omega)$ is determined from the relation $\alpha^2 = i\omega/K_{\text{F}}(\omega)$, Re $\alpha \ge 0$.

The function $\alpha(\omega)$ is analytic with a cut along the beam Re $\omega = 0$, Im $\omega > 0$ [9, 10]. It is easy to calculate the values on the cut shores:

$$\alpha_+ = \alpha\,(iy + \varepsilon) = iy^{1/2}\,(K_{\text{F}+})^{-1/2}\,; \tag{14}$$

$$\alpha_- = \alpha\,(iy - \varepsilon) = -\,iy^{1/2}\,(K_{\text{F}-})^{-1/2}\,. \tag{15}$$

Problem (12)-(13) has the following solution:

$$\Phi_{\text{F}} = A_0 K_0\,(\alpha r) + A_1 I_0\,(\alpha r)\,, \tag{16}$$

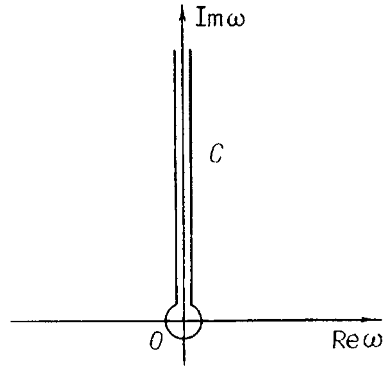

Fig. 1. Contour of integration for integral (18).

$$A_0 = G_1^{-1} \psi I_0(\alpha r_2), \quad A_1 = -G_1^{-1} \psi K_0(\alpha r_2),$$

$$G_1 = I_0(\alpha r_2)(i\omega K_0(\alpha) + \xi K_F \alpha K_1(\alpha)) -$$

$$- K_0(\alpha r_2)(i\omega I_0(\alpha) - \xi K_F \alpha I_1(\alpha)),$$

$$\psi = \eta (i\omega + \varepsilon)^{-1},$$

where $K_n(z)$ and $I_n(z)$ are the Macdonald functions [13].

We will seek an intermediate asymptotics for the LRC, when the effect of finiteness of the supply contour radius r_2 is insignificant. Letting r_2 in Eq. (17) go to infinity and using asymptotic forms for the Macdonald functions [13], after fulfillment of the inverse Fourier transformation, for the function φ we obtain the expression:

$$\varphi(t) = \eta (2\pi)^{-1} \int (i\omega + \varepsilon)^{-1} \exp(i\omega t) f_1(\omega)\, d\omega, \tag{17}$$

$$f_1(\omega) = K_0(\alpha)(i\omega K_0(\alpha) + \xi K_F \alpha K_1(\alpha))^{-1}.$$

Formula (17) represents $\varphi(t)$ in the form of a functional of the kernel K. We will seek the leading asymptotics of this functional at large times t, which however are assumed to be comparable with internal relaxation times. To do this, it is necessary, according to the procedure of [9, 10], to leave in expression (17) the leading terms in the limit $\omega \to 0$, but this limit must not be taken for an argument of the Fourier transform of the kernel. In addition, it is necessary to leave a contribution related to the finiteness of the well volume, since the direct transition $\omega \to 0$ leads to an asymptotics that coincides formally with that for the PRC [10].

After the indicated transformations, we obtain the expression

$$\varphi(t) = \eta (2\pi)^{-1} \int (i\omega + \varepsilon)^{-1} \exp(i\omega t) f_2(\omega)\, d\omega, \tag{18}$$

$$f_2(\omega) = 2^{-1} \ln(i\omega)(2^{-1} i\omega \ln(i\omega) - \xi K_F)^{-1}.$$

Now we transform the integral over the real axis in formula (18) into an integral over the contour C (see Fig. 1) with allowance for Eqs. (7), (14), (15). Resolving the integrands, we derive:

$$\varphi(t) \approx -(2\pi)^{-1} \eta i (i\pi \ln \varepsilon + I_{1\varepsilon} + I_{2\varepsilon}), \tag{19}$$

$$I_{1\varepsilon} = i\pi \int\limits_{\varepsilon}^{+\infty} y^{-1} \exp(-yt)\, dy, \quad I_{2\varepsilon} = \int\limits_{\varepsilon}^{+\infty} y^{-1} (f_2(iy + \varepsilon) - f_2(iy - \varepsilon) - i\pi) \exp(-yt)\, dy.$$

Here ε is the radius of the infinitely small circle along which the point $\omega = 0$ passes (see Fig. 1). To pass to the limit $\varepsilon \to 0$, in formula (20) we must use two additional formulas from [14], namely, formula No. 3.352.4:

$$\int\limits_0^{+\infty} \frac{\exp(-bz)\, dz}{a + z} = -\exp(ab)\ \mathrm{Ei}(-ab) \quad (a, b > 0); \tag{20}$$

and formula No. 8.214.1

$$\mathrm{Ei}(z) = \mathrm{C} + \ln(-z) + \sum_{n=1}^{\infty} z^n (n\, n!)^{-1} \quad (z < 0). \tag{21}$$

We note that the integral $I_{2\varepsilon}$ converges for $\varepsilon \to 0$. When $\varepsilon \to 0$, the limit $I_{1\varepsilon}$ is calculated from formulas (20) and (21). As a result, expression (19) takes a form that is free of the parameter ε:

$$\varphi(t) = (2\pi)^{-1} \eta (\pi \ln t + \pi \ln \mathrm{C} + iI_{20}). \tag{22}$$

We write the principal term of the asymptotics for I_{20}

$$I_{20} \approx (-i \ln tJ(t)), \tag{23}$$

$$J(t) = \xi^{-1} \pi \int\limits_0^{+\infty} y^{-1} (y^{-1} A(y^{-1}) + \xi^{-1} y) \left| \xi^{-1} y (\ln y + i\pi) - K_{\Gamma+} \right|^{-2} \exp(-yt)\, dy.$$

Formulas (22) and (23) give a solution in general form for the problem of the LRC. However, practical applications on interpretation of experimental LRC require a specific form of the function $J(t)$. The asymptotics of $J(t)$ at large t is determined by the asymptotics of the weight function $A(\tau)$ at large relaxation times τ. Suppose that at large τ there is a power spectrum

$$A(\tau) \approx a_0 \tau^{-1-\beta}, \quad 0 < \beta < 1. \tag{24}$$

Assumption (24) is consistent with the convergence condition of integral (6). From Eqs. (23) and (24) we find the asymptotics of $J(t)$ at large t:

$$J(t) \approx \pi\xi^{-1} (a_1 t^{-\beta} + \xi^{-1} t^{-1}), \quad a_1 = a_0 \Gamma(\beta). \tag{25}$$

where $\Gamma(z)$ is a gamma-function [15].

Substituting asymptotics (25) into Eq. (23), we obtain a formula for the LRC for the power spectrum of internal relaxation times. As compared to the asymptotics for classical Darcy law (corresponding to the case $a_0 = 0$), this formula contains two additional fitting parameters, namely, β and a_1. Therefore, in principle, by means of the LRC it is possible to determine simultaneously the permeability of a collector and the relaxational characteristics.

NOTATION

t, t_0, time; x^i, x^j, Cartesian coordinates; ω, frequency; u^i, velocity of filtration; k, permeability; m, porosity; μ, shear viscosity of fluid; p, p_{bed}, pressure; ρ, mass density; U, gravitational potential; $A = A(\tau)$, weight function; k_1, a_0, a_1, β, parameters that characterize the relaxational kernel; E_1 and E_2, volume elasticity modulus of fluid and rock skeleton; $E = (E_1^{-1} + (m^{-1} + (m^{-1} - 1)E_2^{-1})^{-1}$; r, distance from the well axis; Δ, Laplace operator; r_1, radius of the well bottom; r_2, radius of the supply contour; $q = q(t)$, mass inflow of liquid from the collector to the well; h, thickness of productive layer; S, area of the effective cross section of the well; g, free fall acceleration; y, z, λ, ν, ξ, Q_0, Q_1, A_0, A_1, auxiliary parameters; L_1, L_2, Φ, φ, α, ψ, f_1, f_2, $I_{1\varepsilon}$, $I_{2\varepsilon}$, auxiliary functions; C, Euler constant.

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